ON THE HÖRMANDER CLASSES OF BILINEAR PSEUDODIFFERENTIAL OPERATORS II

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Abstract. Boundedness properties for pseudodifferential operators with symbols in the bilinear Hörmander classes of sufficiently negative order are proved. The results are obtained in the scale of Lebesgue spaces and, in some cases, end-point estimates involving weak-type spaces and BMO are provided as well. From the Lebesgue space estimates, Sobolev ones are then easily obtained using functional calculus and interpolation. In addition, it is shown that, in contrast with the linear case, operators associated with symbols of order zero may fail to be bounded on product of Lebesgue spaces.

1. Introduction

In this article we continue the systematic study of the general Hörmander classes of bilinear pseudodifferential operators $BS^m_{\rho,\delta}$ (see the next section for definitions) started in [2]. While the work in [2] focussed mainly on basic properties related to the symbolic calculus of the bilinear pseudodifferential operators and some point-wise estimates for their kernels, the present work addresses boundedness properties on the full scale of Lebesgue spaces. The general properties developed in [2] will become very useful in this current work and will allow us to provide a fairly complete range of results.

The literature on bilinear pseudodifferential operators continues to grow and [2] gives also a historical account and motivations, as well as numerous references in the subject. We would like to reiterate here that most results so far have dealt with the cases $\rho = 1$ and $\rho = 0$. For the first value of ρ the available boundedness and unboundedness results, and other properties of the classes $BS^0_{1,\delta}$ are similar to the ones in the linear situation. They are closely tied to the (bilinear) Calderón-Zygmund theory, which was started by Coiman-Meyer in the 70's (see e.g. [13] and the references therein) and was further developed by Christ-Journé [11], Kenig-Stein [21] and Grafakos-Torres [15]. See also Bényi-Torres [3] and Maldonado-Naibo [23]. The value of $\rho = 0$, however, produces some surprises and the possible theory deviates from the linear situation. In particular the famous Calderón-Villancourt theorem

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[10] does not hold unless further properties on the symbols in $BS^0_{0,0}$ are imposed; see Bényi-Torres [4] and Bernicot-Shrivastava [9].

One important contribution for other values of ρ, almost the exception so far, is the recent work of Michalowski-Rule-Staubach [24]. Since, for example, the class $BS^0_{0,0}$ does not map $L^\infty \times L^2 \to L^2$, it was asked in [2] (and some answers were provided) about results of the form $X \times L^2 \to L^2$ with some functional space X smaller than L^∞ and symbols in $BS^0_{\rho,\delta}$. The question of whether the classes $BS^0_{\rho,\delta}$ produce operators that are bounded on some product of Lebesgue spaces when $0 \leq \delta < \rho$ was left unanswered in [2] (recall the keystone result that the linear class $S^0_{\rho,\delta}$ is bounded on L^2, as proved by Hörmander [17]). Likewise in [24] the authors asked about which negative values of $m = m(\rho)$ produce classes $BS^m_{\rho,\delta}$ for which the corresponding bilinear pseudodifferential operators are bounded from $L^{p_1} \times L^{p_2}$ into L^p with $1/p_1 + 1/p_2 = 1/p$ and $1 < p_1, p_2, p \leq \infty$. Here, we will expand and improve some of the results in [24] in several directions.

First, we will show that it is very much relevant to look at negative values of m when $\rho < 1$ because operators with symbols in the classes $BS^0_{\rho,\delta}$ may fail to be bounded on any product of Lebesgue spaces. This is proved in Theorem 1 below, thus answering in the negative the question left unanswered in [2]. Next we show in Theorem 2 that the values of m provided in [24] can be taken much larger (smaller in absolute value). We succeed in doing so using kernel estimates and the symbolic calculus from [2], also used in [24], but adding arguments involving the complex interpolation of the classes $BS^m_{\rho,\delta}$. Moreover, bringing back the bilinear Calderón-Zygmund theory for sufficiently negative values of m and using further interpolation arguments we also obtain results outside the *Banach triangle*; i.e., for $1/p_1 + 1/p_2 = 1/p$, but $1/2 < p < 1$. We also obtain appropriate weak-type end-point estimates at one end and a strong one at another. This last is the bilinear analog of a result of C. Fefferman, which was also a keystone in the understanding of linear pseudodifferential operators.

Fefferman [14] showed, in particular, that the linear classes $S^{-(1-\rho)\frac{n}{2}}_{\rho,0}$, for $0 < \rho < 1$, map $L^\infty \to BMO$. The natural conjecture then is that $BS^{-(1-\rho)n}_{\rho,0}$ should map $L^\infty \times L^\infty \to BMO$, since often the role of n in the linear case is played by $2n$ in the bilinear setting. We are able to prove this conjecture in Theorem 4 at least for $0 < \rho < 1/2$. Though we use some ideas from [14], new technical difficulties not present in the linear case need to be overcome. In fact, Fefferman used the result of Hörmander that operators with symbols in $S^0_{\rho,\delta}$ are bounded on L^2 but, as Theorem 1 establishes, the analogous result for bilinear operators is false. Instead we rely on the $L^2 \times L^2 \to L^2$ boundedness of certain classes of symbols as proved in Theorem 3.

The article is organized as follows. In the next section we include the main definitions, some basic properties and the precise statements of the main theorems. We also provide some further motivation and applications. The subsequence sections, Sections 3-7, contain the detailed proof of each of the main theorems in the order we list them, except that a series of technical lemmata used in the proof of Theorem 4 are postponed until Section 8. Section 9 contains some weighted versions of the results. Further remarks about the results and comparisons to other linear and bilinear ones are provided throughout the paper as well. Upper-case letters are used

to label theorems corresponding to known results while single numbers are used for theorems, lemmas and corollaries that are proved in this article.

Unless otherwise indicated, the underlying space for the functional classes used will be Euclidean space $\mathbb{R}^n$. In particular, L^p will stand for $L^p(\mathbb{R}^n)$ and $W^{s,p}$ will stand for $W^{s,p}(\mathbb{R}^n)$, the Sobolev space of functions with "s derivatives" in L^p. Their respective norms will be denoted $\|f\|_{L^p}$ and $\|f\|_{W^{s,p}}$. Finally, $\mathcal{S}$ will indicate the Schwartz class on $\mathbb{R}^n$.

Throughout the symbol $\lesssim$ will be used in inequalities where constants are independent of its left and right hand sides.

2. Main Results

Let δ, $\rho \geq 0$ and $m \in \mathbb{R}$. In [17], Hörmander introduced the class of symbols $S^m_{\rho,\delta}$: $\sigma = \sigma(x,\xi)$, x, $\xi \in \mathbb{R}^n$, belongs to $S^m_{\rho,\delta}$ if for all multi-indices α and β

$$\sup_{x,\,\xi\in\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|(1+|\xi|)^{-m-\delta|\alpha|+\rho|\beta|} < \infty.$$

For each symbol σ there is an associated linear pseudodifferential operator T_σ defined by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi, \qquad f \in \mathcal{S},$$

where $\widehat{f}$ denotes the Fourier transform of f.

The bilinear counterpart of $S^m_{\rho,\delta}$ is denoted $BS^m_{\rho,\delta}$. A bilinear symbol $\sigma(x,\xi,\eta)$, x, ξ, $\eta \in \mathbb{R}^n$, belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for all multi-indices α, β and γ,

$$\sup_{x,\xi,\eta\in\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)|(1+|\xi|+|\eta|)^{-m-\delta|\alpha|+\rho(|\beta|+|\gamma|)} < \infty.$$

For $\sigma \in BS^m_{\rho,\delta}$ and non-negative integers K and N define

$$\|\sigma\|_{K,N} := \sup_{\substack{|\alpha|\leq K \\ |\beta|,|\gamma|\leq N}} \sup_{x,\xi,\eta\in\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)|(1+|\xi|+|\eta|)^{-m-\delta|\alpha|+\rho(|\beta|+|\gamma|)}.$$

Then the family of norms $\{\|\cdot\|_{K,N}\}_{K,N\in\mathbb{N}_0}$ turns $BS^m_{\rho,\delta}$ into a Fréchet space.

For $\sigma \in BS^m_{\rho,\delta}$ we consider the bilinear pseudodifferential operator defined by

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\, e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta, \quad f,\, g \in \mathcal{S}.$$

We know proceed to state the new results in this article.

Theorem 1. *Let $0 \leq \rho < 1$, $0 \leq \delta \leq 1$, and $1 \leq p, p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. There exist symbols in $BS^0_{\rho,\delta}$ that give rise to unbounded operators from $L^{p_1} \times L^{p_2}$ into L^p.*

As mentioned in the introduction, the result in Theorem 1 is in contrast with the fact that linear pseudodifferential operators of order zero do produce bounded operators on L^2. The case $\rho = \delta = 0$ of Theorem 1 was proved by Bényi and Torres in [4].

Theorem 2. *Let* $0 \le \delta \le \rho \le 1$, $\delta < 1$, $1 \le p_1, p_2 \le \infty$, p *given by* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$m < m(p_1, p_2) := n(\rho - 1)\left(\max\{\tfrac{1}{2}, \tfrac{1}{p_1}, \tfrac{1}{p_2}, 1 - \tfrac{1}{p}\} + \max\{\tfrac{1}{p} - 1, 0\}\right),$$

and $\sigma \in BS^m_{\rho,\delta}$.

(i) If $p \ge 1$ *then there exist* K, $N \in \mathbb{N}_0$ *such that*

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}}\, \|g\|_{L^{p_2}}$$

for all $f \in L^{p_1}$ *and* $g \in L^{p_2}$.

(ii) If $0 < \rho$, $p < 1$, $p_1 \ne 1$ *and* $p_2 \ne 1$ *then there exist* K, $N \in \mathbb{N}_0$ *such that*

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}}\, \|g\|_{L^{p_2}}$$

for all $f \in L^{p_1}$ *and* $g \in L^{p_2}$.

(iii) If $0 < \rho$, $p < 1$ *and* $p_1 = 1$ *or* $p_2 = 1$ *then then there exist* K, $N \in \mathbb{N}_0$ *such that*

$$\|T_\sigma(f,g)\|_{L^{p,\infty}} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}}\, \|g\|_{L^{p_2}}$$

for all $f \in L^{p_1}$ *and* $g \in L^{p_2}$.

When $p \ge 1$ (Banach case), Theorem 2 improves the results in [24, Theorem 5.5] by Michalowski, Rule and Staubach which require $m < n(\rho-1)\max\{\frac{1}{2}, (\frac{2}{p_1} - \frac{1}{2}), (\frac{2}{p_2} - \frac{1}{2}), (\frac{3}{2} - \frac{2}{p})\}$. This improvement is based on the following facts:

(1) Bilinear pseudodifferential operators with symbols in the classes $BS^m_{\rho,\delta}$ with $m < n(\rho - 1)$ (as opposed to $m < \frac{3}{2}n(\rho - 1)$ used in [24]) are bounded from $L^\infty \times L^\infty$ into L^∞, with norm bounded by the norm of the symbol (see also Remark 4.1).
(2) Roughly speaking, the intermediate spaces in the complex interpolation of two bilinear Hörmander classes are other bilinear Hörmander classes.

When $p < 1$ (non-Banach case), the result of Theorem 2 relies on interpolation arguments using boundedness of operators in the Banach case and bilinear Calderón-Zygmund theory.

We remark that the operator T_σ is a priori defined on $\mathcal{S}\times\mathcal{S}$. In Theorem 2, $T_\sigma(f,g)$ for $f \in L^{p_1}$ and $g \in L^{p_2}$ denotes the "value" given by a bounded extension of the operator, which exists and is unique in the cases $p_1 < \infty$ and $p_2 < \infty$, and is shown to exist when $p_1 = \infty$ or $p_2 = \infty$.

Theorem 3. *If* $\sigma(x,\xi,\eta)$, $x,\xi,\eta \in \mathbb{R}^n$, *is a bilinear symbol such that*

$$C(\sigma) := \sup_{\substack{|\beta| \le [\frac{n}{2}]+1 \\ |\alpha| \le 2(2n+1)}} \sup_{\xi, y \in \mathbb{R}^n} \|\partial_\xi^\alpha \partial_y^\beta \sigma(y, \xi - \cdot, \cdot)\|_{L^2} < \infty,$$

then T_σ *maps continuously* $L^2 \times L^2$ *into* L^2 *with*

$$\|T_\sigma\|_{L^2\times L^2 \to L^2} \lesssim C(\sigma).$$

Theorem 4. *If* $\sigma \in BS^{n(\rho-1)}_{\rho,0}$, $0 \le \rho < \frac{1}{2}$, *then there exists* K, $N \in \mathbb{N}_0$ *such that*

$$\|T_\sigma(f,g)\|_{BMO} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^\infty}\, \|g\|_{L^\infty}, \quad f,\, g \in \mathcal{S}.$$

Theorem 4, which complements the endpoint $m = n(\rho - 1)$ for $p_1 = p_2 = \infty$ in Theorem 2, can be thought of as a bilinear counterpart (when $0 \le \rho < \frac{1}{2}$ and $\delta = 0$) to the following linear result proved by C. Fefferman in [14].

Theorem A (Fefferman [14])**.** *If σ is a symbol in the linear Hörmander class $S^{-\frac{n}{2}(1-\rho)}_{\rho,\delta}$ with $0 \le \delta < \rho < 1$, then T_σ maps L^∞ continuously into BMO.*

The proof of Theorem A uses the fact that the linear class $S^0_{\rho,\delta}$, $0 < \delta < \rho \le 1$, maps L^2 continuously into L^2. The bilinear counterpart of this result is false by Theorem 1. Our proof of Theorem 4 relies on Fefferman's ideas and the result given by Theorem 3.

Next, we present a result concerning boundedness properties of bilinear pseudodifferential operators on Lebesgue spaces with indices that satisfy the Sobolev scaling, as opposed to the Hölder scaling employed in the previous theorems.

Theorem 5. *Let $0 \le \delta \le 1$, $0 < \rho \le 1$, $s \in (0, 2n)$, and $m_s := 2n(\rho - 1) - \rho s$. If $\sigma \in BS^m_{\rho,\delta}$, $m \le m_s$, $1 < p_1, p_2 < \infty$, and $q > 0$ is given by $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_1} - \frac{s}{n}$, then there exist K, $N \in \mathbb{N}$ such that*

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \|\sigma\|_{K,N} \, \|f\|_{L^{p_1}} \, \|g\|_{L^{p_2}}$$

for all $f \in L^{p_1}$ and $g \in L^{p_2}$.

We end this section by briefly featuring some remarks, motivations and applications in the next three subsections.

2.1. **The operator norm, the number of derivatives, and complex interpolation of the classes of symbols.** Theorems 2 and 5 state that the operator norm of T_σ, as a bounded operator from a product of Lebesgue spaces into another Lebesgue space, is controlled by $\|\sigma\|_{K,N}$ for some nonnegative integers K and N. Even though this is a consequence of the proof provided in each case, it can be shown to be a necessary condition. More precisely,

Lemma 6. *Let $0 < p \le \infty$, $1 \le p_1, p_2 < \infty$, $0 \le \delta$, $\rho \le 1$ and suppose T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $\sigma \in BS^m_{\rho,\delta}$. Then there exist K, $N \in \mathbb{N}_0$ such that*

$$\|T_\sigma\| \lesssim \|\sigma\|_{K,N} \quad \textit{for all } \sigma \in BS^m_{\rho,\delta}.$$

Indeed, Lemma 6 is a consequence of the Closed Graph Theorem. Consider in $BS^m_{\rho,\delta}$ the topology induced by the family of norms $\{\|\cdot\|_{K,N}\}_{K,N\in\mathbb{N}_0}$, as defined above, which turns $BS^m_{\rho,\delta}$ into a Fréchet space. If T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $\sigma \in BS^m_{\rho,\delta}$ we can define the linear transformation

$$U : BS^m_{\rho,\delta} \to \mathcal{L}(L^{p_1} \times L^{p_2}, L^p), \qquad U(\sigma) = T_\sigma,$$

where $\mathcal{L}(L^{p_1} \times L^{p_2}, L^p)$ denotes the quasi-Banach space (Banach space if $p \ge 1$) of all bilinear bounded operators from $L^{p_1} \times L^{p_2}$ into L^p endowed with the operator quasi-norm (norm if $p \ge 1$). If $\{(\sigma_k, T_{\sigma_k})\}_{k\in\mathbb{N}}$ is a sequence in the graph of U that converges to (σ, T), for some $\sigma \in BS^m_{\rho,\delta}$ and $T \in \mathcal{L}(L^{p_1} \times L^{p_2}, L^p)$, then it easily follows that $T(f, g) = T_\sigma(f, g)$ for any $f, g \in \mathcal{S}(\mathbb{R}^n)$. Since T_σ and T are bilinear bounded operators from $L^{p_1} \times L^{p_2}$ into L^p, by density, we obtain that $T = T_\sigma$. Then

the graph of U is closed and therefore, by the closed Graph Theorem, U is continuous and the desired result follows.

In regards to the number of derivates required for the symbols, we remark that the following modified versions of the bilinear Hörmander classes can be considered: For $K,\, N \in \mathbb{N}_0$,

$$BS^m_{\rho,\delta,K,N} := \{\sigma(x,\xi,\eta) \in C^{K,N}(\mathbb{R}^{3n}) : \|\sigma\|_{K,N} < \infty\},$$

where $C^{K,N}(\mathbb{R}^{3n})$ means derivatives up to order K in x and up to order N in ξ and η. Then $BS^m_{\rho,\delta,K,N}$ is a Banach space with norm $\|\cdot\|_{K,N}$ that contains $BS^m_{\rho,\delta}$ as a dense subset and therefore the results of Theorem 2, 4, and 5 remain true if $BS^m_{\rho,\delta}$ is replaced with $BS^m_{\rho,\delta,K,N}$ for appropriate values of $K,\, N \in \mathbb{N}_0$, possibly depending on m, ρ, and δ. We will not pursue in this paper the question regarding the minimum number of derivatives needed to achieve the results presented, though some estimates can be inferred from the proofs.

We close this subsection with a result on the complex interpolation of the classes $BS^m_{\rho,\rho,N,N}$ which will be useful in the proof of Theorem 2.

Lemma 7. *If $m_0, m_1 \in \mathbb{R}$, $0 \le \rho < 1$ and $m = \theta\, m_0 + (1-\theta)\, m_1$ for some $\theta \in (0,1)$ then*

$$\left(BS^{m_0}_{\rho,\rho,N,N}, BS^{m_1}_{\rho,\rho,N,N}\right)_{[\theta]} = BS^m_{\rho,\rho,N,N}.$$

Indeed, the lemma follows using the same arguments as in the work of Päivärinta-Somersalo [27, Lemma 3.1], where the analogous result for the linear Hörmander classes is proved.

2.2. **Leibniz-type rules.** In terms of applications of the bilinear L^p-theory for the class $BS^m_{\rho,\delta}$, the results in this paper allow for enriched versions of the fractional Leibniz rule

$$\|fg\|_{W^{s,p}} \le C\left(\|f\|_{W^{s,p_1}}\, \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\, \|g\|_{W^{s,p_2}}\right), \tag{2.1}$$

where $s \ge 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p_1,\, p_2 < \infty$ (see Kato-Ponce [18], Christ-Weinstein [12], and Kenig-Ponce-Vega [19]).

Inequalities of the type (2.1) for pseudodifferential operators $T_\sigma(f,g)$ instead of the product fg $(\sigma \equiv 1)$ can be easily obtained following what is by now a well-known procedure that uses results going back to Coifman and Meyer and has become part of the folklore in the subject. The idea, as already used in [18], is to (smoothly) split the symbol into frequency regions where the derivatives can be distributed among the functions. See also Semmes [28] and Gulisashvili-Kon [16] where both homogeneous and inhomogeneous derivatives were considered in similar fashion.

Consider $\sigma \in BS^m_{\rho,\delta}$ and $\phi \in C^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\mathrm{supp}(\phi) \subset [-2,2]$ and $\phi(r) + \phi(1/r) = 1$ on $[0,\infty)$. For $s > 0$, the symbols σ_1 and σ_2 given by

$$\sigma_1(x,\xi,\eta) = \sigma(x,\xi,\eta)\phi\left(\frac{1+|\eta|^2}{1+|\xi|^2}\right)(1+|\xi|^2)^{-(m+s))/2},$$

$$\sigma_2(x,\xi,\eta) = \sigma(x,\xi,\eta)\phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)(1+|\eta|^2)^{-(m+s))/2},$$

satisfy $\sigma_1, \sigma_2 \in BS^{-s}_{\rho,\delta}$, and the corresponding operators T_σ, T_{σ_1}, and T_{σ_2} are related through

$$T_\sigma(f,g) = T_{\sigma_1}(J^{m+s}f, g) + T_{\sigma_2}(f, J^{m+s}g),$$

where J^{m+s} denotes the linear Fourier multiplier with symbol $(1+|\cdot|^2)^{(m+s)/2}$. Thus, the boundedness properties on Lebesgue spaces of bilinear pseudodifferential operators given in Theorems 2 and 5 imply

$$\|T_\sigma(f,g)\|_{L^p} \le C\left(\|f\|_{W^{m+s,p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|g\|_{W^{m+s,q_2}}\right), \quad f, g \in \mathcal{S}, \tag{2.2}$$

for appropriate values of p_1, p_2, q_1, q_2 and s. We refer the reader to Bernicot et al [8] for additional Leibniz-type rules.

In the same spirit, using the functional rule

$$\partial_{x_i} T_\sigma(f,g) = T_{\partial_{x_i}\sigma}(f,g) + T_\sigma(\partial_{x_i} f, g) + T_\sigma(f, \partial_{x_i} g),$$

the fact that $\sigma \in BS^m_{\rho,\delta}$ yields $\partial_{x_i}\sigma \in BS^{m+\delta}_{\rho,\delta}$, and bilinear complex interpolation, Theorem 2 and Theorem 5 imply the following corollaries:

Corollary 8. *Let $0 \le \delta \le \rho \le 1$, $\delta < 1$, $1 \le p_1, p_2 \le \infty$, p given by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $m(p_1,p_2)$ as in Theorem 2. If $\sigma \in BS^m_{\rho,\delta}$, $m < m(p_1,p_2) - k\delta$ for some nonnegative integer k, and $r \in [0,k]$, then there exists K, $N \in \mathbb{N}_0$ such that*

$$\|T_\sigma(f,g)\|_{W^{r,p}} \lesssim \|\sigma\|_{K,N} \|f\|_{W^{r,p_1}} \|g\|_{W^{r,p_2}},$$

for all $f \in W^{r,p_1}$ and $g \in W^{r,p_2}$.

Corollary 9. *Let $0 \le \delta \le 1$, $0 < \rho \le 1$, $s \in (0,2n)$, $m_s = 2n(\rho-1) - \rho$ as in Theorem 5, $1 < p_1, p_2 < \infty$, and $q > 0$ such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_1} - \frac{s}{n}$. . If $\sigma \in BS^m_{\rho,\delta}$, $m \le m_s - k\delta$, for some nonnegative integer k, and $r \in [0,k]$, then there exists K, $N \in \mathbb{N}_0$ such that*

$$\|T_\sigma(f,g)\|_{W^{r,q}} \lesssim \|\sigma\|_{K,N} \|f\|_{W^{r,p_1}} \|g\|_{W^{r,p_2}},$$

for all $f \in W^{r,p_1}$ and $g \in W^{r,p_2}$.

2.3. **Applications to the scattering of PDEs.** Consider the system of partial differential equations for $u = u(t,x)$, $v = v(t,x)$, and $w = w(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$,

$$\left\{ \begin{array}{ll} \partial_t u + a(D)u = vw, & u(0,x) = 0, \\ \partial_t v + b(D)v = 0, & v(0,x) = f(x), \\ \partial_t w + c(D)w = 0, & w(0,x) = g(x). \end{array} \right. \tag{2.3}$$

where $a(D)$, $b(D)$ and $c(D)$ are linear multipliers with symbols $a(\xi)$, $b(\xi)$ and $c(\xi)$, $\xi \in \mathbb{R}^n$, respectively. Then, formally,

$$v(t,x) = \int_{\mathbb{R}^n} e^{-tb(\xi)} \widehat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi, \quad w(t,x) = \int_{\mathbb{R}^n} e^{-tc(\eta)} \widehat{g}(\eta)\, e^{ix\cdot\eta}\, d\eta,$$

and

$$v(t,x)w(t,x) = \int_{\mathbb{R}^{2n}} e^{-t(b(\xi)+c(\eta))} \widehat{f}(\xi)\widehat{g}(\eta)\, e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

Another formal computation then yields

$$u(t,x) = (e^{-ta(D)} F(t,\cdot))(x),$$

where

$$\begin{aligned} F(t,x) &= \int_0^t e^{sa(D)}(v(s,\cdot)w(s,\cdot))(x)ds \\ &= \int_{\mathbb{R}^{2n}} \left(\int_0^t e^{s(a(\xi+\eta)-b(\xi)-c(\eta))} ds \right) \widehat{f}(\xi)\widehat{g}(\eta)\, e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta. \end{aligned}$$

Therefore, if the phase function $\lambda(\xi,\eta) := a(\xi+\eta) - b(\xi) - c(\eta)$ does not vanish,

$$F(t,x) = T_{\frac{e^{t\lambda}-1}{\lambda}}(f,g)(x).$$

As a consequence, assuming that $\lambda < 0$, the solution u of (2.3) scatters in the Sobolev space $W^{r,p}$ if

$$\lim_{t\to\infty} T_{\frac{e^{t\lambda}-1}{\lambda}}(f,g) = T_{-\lambda^{-1}}(f,g) \in W^{r,p}.$$

According to Corollary 8, $T_{-\lambda^{-1}}$ is a bounded operator on Sobolev spaces if $-\lambda^{-1}$ belongs to $BS^m_{\rho,\delta}$ for suitable exponents.

As an example consider $b(D) = 1 - \Delta$ and $c(D) = |D|$. Then for $a(D) = 0$, we get

$$-\lambda(\xi,\eta)^{-1} = (1 + |\xi|^2 + |\eta|)^{-1}$$

and

$$\lambda(\xi,\eta)^{-1}\varphi(\xi,\eta) \in BS^{-1}_{\frac{1}{2},0},$$

for any smooth function φ such that $\varphi = 1$ away from the set $\{(\xi,\eta) : \eta = 0\}$. In the case that $a(D) = \Delta$, we get

$$-\lambda(\xi,\eta)^{-1} = (1 + |\xi+\eta|^2 + |\xi|^2 + |\eta|)^{-1}$$

and

$$\lambda(\xi,\eta)^{-1}\varphi(\xi,\eta) \in BS^{-2}_{1,0}.$$

When the phase function λ vanishes, the situation is more difficult. We refer the reader to [5, 6], where a more precise study has been developed to obtain bilinear dispersive estimates (instead of scattering properties).

3. Proof of Theorem 1

As we will show, Theorem 1 follows from the case corresponding to $\rho = \delta = 0$, a scaling argument and Lemma 6. We first need to recall the following result.

Theorem B (Bényi-Torres [4, Proposition 1]). *There exist x-independent symbols in $BS^0_{0,0}$ that give rise to unbounded operators from $L^{p_1} \times L^{p_2}$ into L^p for $1 \le p_1, p_2, p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

Proof of Theorem 1. Fix $\delta, \rho, p_1, p_2, p$ as in the hypothesis. Suppose, on the contrary, that T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $\sigma \in BS^0_{\rho,\delta}$.

Consider an x-independent symbol $\sigma \in BS^0_{\rho,\delta}$ and, for multi-indices β, γ, set

$$C_{\beta,\gamma}(\sigma) := \sup_{\xi,\eta\in\mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi,\eta)|(1 + |\xi| + |\eta|)^{\rho(|\beta|+|\gamma|)}.$$

For $\lambda > 0$ define $\sigma_\lambda(\xi,\eta) := \sigma(\lambda\xi,\lambda\eta)$, $\xi,\eta\in\mathbb{R}^n$. Then, for all multi-indices β,γ and $0<\lambda<1$, we have

$$\begin{aligned}|\partial_\xi^\beta\partial_\eta^\gamma\sigma_\lambda(\xi,\eta)| &= \lambda^{|\beta|+|\gamma|}|\partial_\xi^\beta\partial_\eta^\gamma\sigma(\lambda\xi,\lambda\eta)|\\ &\le \lambda^{(1-\rho)(|\beta|+|\gamma|)}C_{\beta,\gamma}(\sigma)(1+|\xi|+|\eta|)^{-\rho(|\beta|+|\gamma|)},\end{aligned}$$

giving

$$C_{\beta,\gamma}(\sigma_\lambda) \le \lambda^{(1-\rho)(|\beta|+|\gamma|)}C_{\beta,\gamma}(\sigma). \tag{3.4}$$

Let $f,\, g\in\mathcal{S}$ and define $f_\lambda(x) := f\left(\frac{x}{\lambda}\right)$ and $g_\lambda(x) := g\left(\frac{x}{\lambda}\right)$, $x\in\mathbb{R}^n$. Then

$$\begin{aligned}T_\sigma(f,g)(x) &= \int_{\mathbb{R}^{2n}}\sigma(\xi,\eta)\hat f(\xi)\hat g(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta\\ &= \int_{\mathbb{R}^{2n}}\sigma\left(\lambda\frac{\xi}{\lambda},\lambda\frac{\eta}{\lambda}\right)\hat f\left(\lambda\frac{\xi}{\lambda}\right)\hat g\left(\lambda\frac{\eta}{\lambda}\right)e^{i\lambda x\cdot\left(\frac{\xi}{\lambda}+\frac{\eta}{\lambda}\right)}\,d\xi d\eta\\ &= \int_{\mathbb{R}^{2n}}\sigma_\lambda(\xi,\eta)\widehat{f_\lambda}(\xi)\widehat{g_\lambda}(\eta)e^{i\lambda x\cdot(\xi+\eta)}\,d\xi d\eta\\ &= T_{\sigma_\lambda}(f_\lambda,g_\lambda)(\lambda x).\end{aligned}$$

Let $K,\, N\in\mathbb{N}_0$ be given by Lemma 6 for the class $BS^0_{\rho,\delta}$ and, without loss of generality, assume $K=N$. Then using that $\|\sigma_\lambda\|_{N,N} = \left(\sup_{|\beta|,\,|\gamma|\le N} C_{\beta,\gamma}(\sigma_\lambda)\right)$, $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$, and (3.4), we obtain

$$\begin{aligned}\|T_\sigma(f,g)\|_{L^p} &= \|T_{\sigma_\lambda}(f_\lambda,g_\lambda)(\lambda\cdot)\|_{L^p} = \lambda^{-\frac{n}{p}}\|T_{\sigma_\lambda}(f_\lambda,g_\lambda)\|_{L^p}\\ &\lesssim \lambda^{-\frac{n}{p}}\left(\sup_{|\beta|,\,|\gamma|\le N} C_{\beta,\gamma}(\sigma_\lambda)\right)\|f_\lambda\|_{L^{p_1}}\|g_\lambda\|_{L^{p_2}}\\ &= \lambda^{-\frac{n}{p}+\frac{n}{p_1}+\frac{n}{p_2}}\left(\sup_{|\beta|,\,|\gamma|\le N} C_{\beta,\gamma}(\sigma_\lambda)\right)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\\ &\lesssim \left(\sup_{|\beta|,\,|\gamma|\le N}\lambda^{(1-\rho)(|\beta|+|\gamma|)}C_{\beta,\gamma}(\sigma)\right)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}},\end{aligned}$$

and letting $\lambda\to 0$, it follows that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim C_{0,0}(\sigma)\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}}\qquad f\in L^{p_1}, g\in L^{p_2}. \tag{3.5}$$

However, (3.5) cannot be true since this contradicts Theorem B. Indeed, take $\sigma\in BS^0_{0,0}$ x-independent such that T_σ is not bounded from $L^{p_1}\times L^{p_2}$ into L^p and let φ be an infinitely differentiable function in $\mathbb{R}^{2n}$ supported in $|(\xi,\eta)|\le 2$ and equal to one on $|(\xi,\eta)|\le 1$. For each $\varepsilon>0$, set $\sigma_\varepsilon(\xi,\eta) := \varphi(\varepsilon\,\xi,\varepsilon\,\eta)\sigma(\xi,\eta)$. Then $\sigma_\varepsilon\in BS^0_{\rho,\delta}(\mathbb{R}^n)$ and $C_{0,0}(\sigma_\varepsilon)\le C_{0,0}(\sigma)$ for all $\varepsilon>0$. If (3.5) were true we would have

$$\|T_{\sigma_\varepsilon}(f,g)\|_{L^p} \lesssim C_{0,0}(\sigma)\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}}\qquad f,\, g\in\mathcal{S},\qquad \text{for all } \varepsilon>0.$$

As $\varepsilon \to 0$, $T_{\sigma_\varepsilon}(f,g) \to T_\sigma(f,g)$ pointwise; this and Fatou Lemma yield

$$\|T_\sigma(f,g)\|_{L^p} \lesssim C_{0,0}(\sigma)\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}} \qquad f,\, g \in \mathcal{S},$$

a contradiction.

□

4. Proof of Theorem 2

4.1. **Preliminary results.** We will use the following results in the proof of Theorem 2.

Theorem C (Symbolic calculus, Bényi-Maldonado-Naibo-Torres [2]). *Assume that $0 \le \delta \le \rho \le 1$, $\delta < 1$, and $\sigma \in BS^m_{\rho,\delta}$. Then, for $j = 1, 2$, $T^{*j}_\sigma = T_{\sigma^{*j}}$, where $\sigma^{*j} \in BS^m_{\rho,\delta}$. Moreover, if $0 \le \delta < \rho \le 1$ and $\sigma \in BS^m_{\rho,\delta}$, then σ^{*1} and σ^{*2} have explicit asymptotic expansions.*

Theorem D (Michalowski-Rule-Staubach [24, Theorem 5.5]). *Let $0 \le \delta \le \rho \le 1$, $\delta < 1$, $1 \le p_1, p_2, p \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and*

$$m < n(\rho-1)\max\{\tfrac{1}{2}, (\tfrac{2}{p_1} - \tfrac{1}{2}), (\tfrac{2}{p_2} - \tfrac{1}{2}), (\tfrac{3}{2} - \tfrac{2}{p})\}.$$

If $\sigma \in BS^m_{\rho,\delta}$, then there exist K, $N \in \mathbb{N}_0$ such that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_{K,N}\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}}\,.$$

Set $\tilde{m}(p_1,p_2) = n(\rho-1)\max\{\frac{1}{2}, (\frac{2}{p_1} - \frac{1}{2}), (\frac{2}{p_2} - \frac{1}{2}), (\frac{3}{2} - \frac{2}{p})\}$ and note that, when $p > 1$, we have $m(p_1,p_2) = n(\rho-1)\max\{\frac{1}{2}, \frac{1}{p_1}, \frac{1}{p_2}, (1 - \frac{1}{p})\}$. Referring to Figure 1, we then have that $m(p_1,p_2) = n(\rho-1)\frac{1}{p_2}$ and $\tilde{m}(p_1,p_2) = n(\rho-1)(\frac{2}{p_2} - \frac{1}{2})$ in region I, $m(p_1,p_2) = n(\rho-1)\frac{1}{p_1}$ and $\tilde{m}(p_1,p_2) = n(\rho-1)(\frac{2}{p_1} - \frac{1}{2})$ in region II, $m(p_1,p_2) = n(\rho-1)(1 - \frac{1}{p})$ and $\tilde{m}(p_1,p_2) = n(\rho-1)(\frac{3}{2} - \frac{2}{p})$ in region III, and $m(p_1,p_2) = \tilde{m}(p_1,p_2) = n(\rho-1)\frac{1}{2}$ in region IV. Then $\tilde{m} < m$ in regions I, II and III, and therefore the Banach case of Theorem 2 is an improvement on Theorem D.

In the non-Banach case ($p < 1$), we will use bilinear Calderón-Zygmund theory to get the boundedness results stated in Theorem 2. Indeed, we have the following result:

Theorem 10 (Bilinear Calderón-Zygmund operators). *Let $0 \le \delta \le \rho \le 1$, $\delta < 1$, $0 < \rho$, and set $m_{cz} := 2n(\rho-1)$. If $\sigma \in BS^m_{\rho,\delta}$ and $m < m_{cz}$, then T_σ is a bilinear Calderón-Zygmund operator. As a consequence, the following mapping properties hold true for $1 \le p_1$, $p_2 \le \infty$, $\frac{1}{2} \le p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$:*

(i) if $1 < p_1$, p_2, then there exist K, $N \in \mathbb{N}_0$ such that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_{K,N}\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}}\,,$$

where L^{p_1} or L^{p_2} should be replaced by L^∞_c (bounded functions with compact support) if $p_1 = \infty$ or $p_2 = \infty$, respectively;

(ii) if $p_1 = 1$ or $p_2 = 1$, then there exist K, $N \in \mathbb{N}$ such that

$$\|T_\sigma(f,g)\|_{L^{p,\infty}} \lesssim \|\sigma\|_{K,N}\,\|f\|_{L^{p_1}}\,\|g\|_{L^{p_2}}\,,$$

where L^{p_1} or L^{p_2} should be replaced by L^∞_c if $p_1 = \infty$ or $p_2 = \infty$, respectively;

(iii) there exist $K,\, N \in \mathbb{N}$ *such that*

$$\|T_\sigma(f,g)\|_{BMO} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^\infty}\, \|g\|_{L^\infty}$$

for $f,\, g \in L^\infty_c$;

(iv) weighted versions of the above inequalities (see Section 9).

The results of Theorem 10 are consequences of the following estimates for the kernel of T_σ:

Theorem E. *Let* $\sigma \in BS^m_{\rho,\delta}$, $0<\rho\le 1$, $0\le\delta<1$, $m\in\mathbb{R}$, *and denote by* $\mathcal{K}(x,y,z)$ *the distributional kernel of the associated bilinear pseudodifferential operator* T_σ. *For* $x,y,z\in\mathbb{R}^n$, *set*

$$S(x,y,z) = |x-y| + |x-z| + |y-z|.$$

(i) Given $\alpha,\beta,\gamma\in\mathbb{N}_0^n$, *there exists* $N_0\in\mathbb{N}_0$ *such that for each* $l\ge N_0$,

$$\sup_{(x,y,z):S(x,y,z)>0} S(x,y,z)^l |D_x^\alpha D_y^\beta D_z^\gamma \mathcal{K}(x,y,z)| < \infty.$$

(ii) Suppose that σ *has compact support in* (ξ,η) *uniformly in* x. *Then* $\mathcal{K}$ *is smooth, and given* $\alpha,\beta,\gamma\in\mathbb{N}_0^n$ *and* $N_0\in\mathbb{N}_0$, *there exists* $C>0$ *such that for all* $x,y,z\in\mathbb{R}^n$ *with* $S(x,y,z)>0$

$$|D_x^\alpha D_y^\beta D_z^\gamma \mathcal{K}(x,y,z)| \le C(1+S(x,y,z))^{-N_0}.$$

(iii) Suppose that $m+M+2n<0$ *for some* $M\in\mathbb{N}_0$. *Then* $\mathcal{K}$ *is a bounded continuous function with bounded continuous derivatives of order* $\le M$.

(iv) Suppose that $m+M+2n=0$ *for some* $M\in\mathbb{N}_0$. *Then there exists a constant* $C>0$ *such that for all* $x,y,z\in\mathbb{R}^n$ *with* $S(x,y,z)>0$,

$$\sup_{|\alpha+\beta+\gamma|=M} |D_x^\alpha D_y^\beta D_z^\gamma \mathcal{K}(x,y,z)| \le C|\log|S(x,y,z)||.$$

(v) Suppose that $m+M+2n>0$ *for some* $M\in\mathbb{N}_0$. *Then, given* $\alpha,\beta,\gamma\in\mathbb{N}_0^n$, *there exists a positive constant* C *such that for all* $x,y,z\in\mathbb{R}^n$ *with* $S(x,y,z)>0$,

$$\sup_{|\alpha+\beta+\gamma|=M} |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \mathcal{K}(x,y,z)| \le C S(x,y,z)^{-(m+M+2n)/\rho}.$$

(vi) Suppose that $m+\varepsilon+2n>0$ *for some* $\varepsilon\in(0,1)$. *Then, there exists a positive constant* C *such that for all* $x,y,z,u\in\mathbb{R}^n$ *with* $S(x,y,z)>0$ *and* $|u|\le S(x,y,z)$,

$$\begin{aligned}|\mathcal{K}(x,y,z)-\mathcal{K}(x+u,y,z)| &+ |\mathcal{K}(x,y,z)-\mathcal{K}(x,y+u,z)|\\ &+|\mathcal{K}(x,y,z)-\mathcal{K}(x,y,z+u)| \le C|u|^\varepsilon S(x,y,z)^{-(m+\varepsilon+2n)/\rho}.\end{aligned}$$

All constants in the above inequalities depend linearly on $\|\sigma\|_{K,N}$ *for some* $K,\, N\in\mathbb{N}_0$.

We refer the reader to [2, Theorem 6] for the proofs of items (i)-(v) in Theorem E. Item (vi) corresponds to the “Hölder” version of item (v), its proof is analogous and relies on estimates for linear kernels as presented in Alvarez-Hounie [1, Theorem 1.1].

Proof of Theorem 10. It is enough to prove the result for $\sigma \in BS^m_{\rho,\delta}$ and m such that $2n(\rho-1)-t < m < 2n(\rho-1) = m_{cz}$ for some small positive number t. Denote by $\mathcal{K}(x,y,z)$ the distributional kernel of the associated bilinear pseudodifferential operator T_σ. Using that $BS^m_{\rho,\delta} \subset BS^{m_{cz}}_{\rho,\delta}$, part (v) of Theorem E applied to $BS^{m_{cz}}_{\rho,\delta}$ yields, with constants depending linearly on $\|\sigma\|_{N,N}$ for some $N \in \mathbb{N}_0$,

$$|\mathcal{K}(x,y,z)| \lesssim \frac{1}{(|x-y|+|x-z|+|y-z|)^{2n}},$$

while part (vi) gives, again with constants depending linearly on $\|\sigma\|_{N,N}$ for some $N \in \mathbb{N}_0$,

$$\begin{aligned}|\mathcal{K}(x,y,z)-\mathcal{K}(x+u,y,z)| + |\mathcal{K}(x,y,z)-\mathcal{K}(x,y+u,z)|&\\ +|\mathcal{K}(x,y,z)-\mathcal{K}(x,y,z+u)| \lesssim \frac{|u|^\varepsilon}{(|x-y|+|x-z|+|y-z|)^{2n+\varepsilon}},&\end{aligned}$$

where $|u| \leq |x-y|+|x-z|+|y-z|$ and $\varepsilon \in (0,1)$ has been chosen such that $(m+2n+\varepsilon)/\rho = 2n+\epsilon$ (which is possible since $2n(\rho-1)-t<m<2n(\rho-1)$ for small enough $t>0$). Moreover, since $m < m_{cz} < n(\rho-1)/2$, Theorem D yields that there exists $N \in \mathbb{N}_0$ such that T_σ satisfies

$$\|T_\sigma(f,g)\|_{L^1} \lesssim \|\sigma\|_{N,N} \|f\|_{L^2} \|g\|_{L^2}.$$

We then conclude that T_σ is a bilinear Calderón-Zygmund operator for which the corresponding boundedness properties follow (see [15]). $\square$

4.2. **Proof of Theorem 2.** With these preliminary and technical results, we are now ready for the proof of our main result in this section.

Proof of Theorem 2. We first prove the theorem for $p_1 = p_2 = p = \infty$, in which case $m(p_1,p_2) = n(\rho-1)$. Let $m < n(\rho-1)$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. Let $\{\psi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity on $\mathbb{R}^{2n}$,

$$\sum_{j=0}^{\infty} \psi_j(\xi,\eta) = 1, \quad \xi,\eta \in \mathbb{R}^n,$$

such that ψ_0 is supported in $\{(\xi,\eta)\in\mathbb{R}^{2n} : |(\xi,\eta)| \leq 2\}$ and $\psi_j(\xi,\eta) = \psi(2^{-j}\xi, 2^{-j}\eta)$, for some $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n})$ supported in $\{(\xi,\eta)\in\mathbb{R}^{2n} : |(\xi,\eta)| \sim 1\}$ for $j \in \mathbb{N}$. We decompose the symbol $\sigma(x,\xi,\eta)$ as

$$\sigma(x,\xi,\eta) = \sum_{j=0}^{\infty} \sigma_j(x,\xi,\eta),$$

where $\sigma_j(x,\xi,\eta) := \sigma(x,\xi,\eta)\psi_j(\xi,\eta)$. Then $\|\sigma_j\|_{0,N} \lesssim \|\sigma\|_{0,N}$ for all $N \in \mathbb{N}_0$ and, by Lemma 11 (see Section 6),

$$\|T_{\sigma_j}(f,g)\|_\infty \lesssim \|\sigma\|_{0,2N}\, 2^{j(m+n(1-\rho))} \|f\|_{L^\infty}\|g\|_{L^\infty}, \quad j \in \mathbb{N}_0,\ N>n,\ K \in \mathbb{N}_0.$$

Therefore

$$\|T_\sigma(f,g)\|_{L^\infty} \leq \sum_{j=0}^{\infty} \|T_{\sigma_j}(f,g)\|_\infty \lesssim \|\sigma\|_{0,2N} \sum_{j=0}^{\infty} 2^{j(m+n(1-\rho))} \|f\|_{L^\infty} \|g\|_{L^\infty}$$
$$\lesssim \|\sigma\|_{0,2N} \|f\|_{L^\infty} \|g\|_{L^\infty},$$

where we have used that $m < n(\rho-1)$. This proves the theorem for $p_1 = p_2 = p = \infty$. Note that the proof shows that there is an extension of T_σ that is bounded from $L^\infty \times L^\infty$ into L^∞, mainly

$$T_\sigma(f,g) = \sum_{j=1}^{\infty} T_{\sigma_j}(f,g)$$

where

$$T_{\sigma_j}(f,g)(x) = \int_{\mathbb{R}^{2n}} \mathcal{K}_j(x, x-y, x-z) f(y) g(z)\, dydz,$$

with

$$\mathcal{K}_j(x,y,z) = \int_{\mathbb{R}^{2n}} \sigma_j(x,\xi,\eta)\, e^{i\xi\cdot y}\, e^{i\eta\cdot z}\, d\xi d\eta, \quad x,y,z \in \mathbb{R}^n.$$

We now proceed to prove the theorem in the general case. We recall that the boundedness properties in Lebesgue spaces for operators corresponding to the class $BS^0_{1,\delta}$ for $0 \leq \delta < 1$ are well-known (see introduction); therefore we will work with $\rho < 1$. Moreover, since $BS^m_{\rho,\delta} \subset BS^m_{\rho,\rho}$ for $\delta \leq \rho$, we will assume $\delta = \rho$, $0 \leq \rho < 1$. Define on $BS^m_{\rho,\rho} \times L^{p_1} \times L^{p_2}$ the trilinear operator T given by

$$T(\sigma, f, g) := T_\sigma(f,g).$$

In the following we will use the notation $T : BS^m_{\rho,\rho} \times X \times Y \to Z$ to express the fact that T maps continuously from $BS^m_{\rho,\rho} \times X \times Y$ into Z : there exists $N \in \mathbb{N}_0$, possibly depending on m and ρ, such that

$$\|T(\sigma,f,g)\|_Z \lesssim \|\sigma\|_{N,N} \|f\|_X \|g\|_Y,$$

for all $\sigma \in BS^m_{\rho,\rho}$, $f \in X$, $g \in Y$.

We first prove (i) (case $p > 1$). The case $p_1 = p_2 = p = \infty$ proved above and Theorem C yield

- $T : BS^m_{\rho,\rho} \times L^\infty \times L^\infty \to L^\infty$ for $m < n(\rho-1)$ (point $(0,0)$ in Figure 1),
- $T : BS^m_{\rho,\rho} \times L^1 \times L^\infty \to L^1$ for $m < n(\rho-1)$ (point $(1,0)$ in Figure 1),
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^1 \to L^1$ for $m < n(\rho-1)$ (point $(0,1)$ in Figure 1).

Moreover, by Theorem D we have

- $T : BS^m_{\rho,\rho} \times L^2 \times L^2 \to L^1$ for $m < \frac{n}{2}(\rho-1)$ (point $(\frac{1}{2},\frac{1}{2})$ in Figure 1),
- $T : BS^m_{\rho,\rho} \times L^2 \times L^\infty \to L^2$ for $m < \frac{n}{2}(\rho-1)$ (point $(\frac{1}{2},0)$ in Figure 1),
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^2 \to L^2$ for $m < \frac{n}{2}(\rho-1)$ (point $(0,\frac{1}{2})$ in Figure 1).

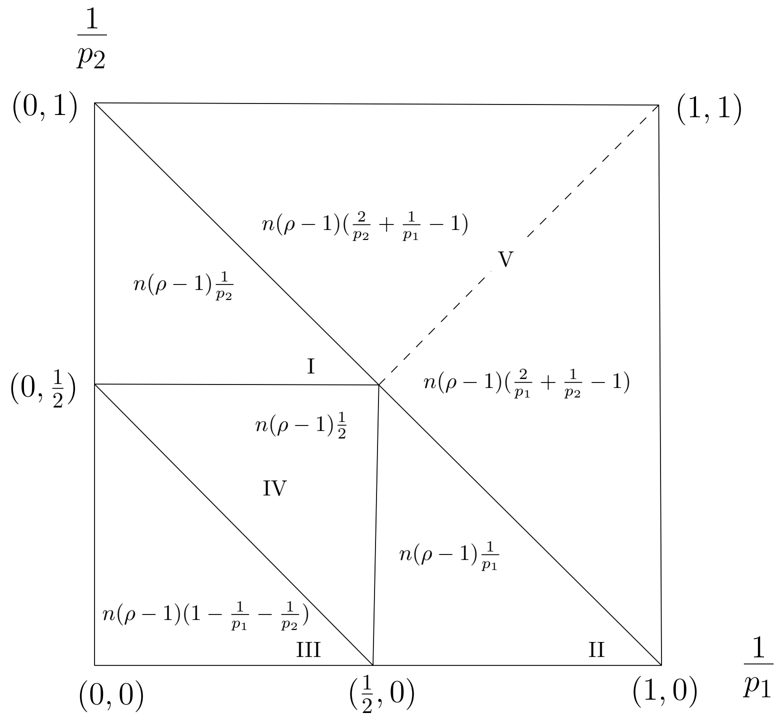

FIGURE 1. Value of $m(p_1, p_2)$ as given by Theorem 2

We now recall the following modified version of the bilinear Hörmander classes (see Section 2.1):

$$BS^m_{\rho,\rho,N,N} := \{\sigma(x,\xi,\eta) \in C^N(\mathbb{R}^{3n}) : \|\sigma\|_{N,N} < \infty\}$$

where $N \in \mathbb{N}_0$ and, as always,

$$\|\sigma\|_{N,N} := \sup_{\substack{|\alpha|\le N\\ |\beta|,|\gamma|\le N}} \sup_{x,\xi,\eta\in\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)|(1+|\xi|+|\eta|)^{-m-\rho|\alpha|+\rho(|\beta|+|\gamma|)}.$$

Since $BS^m_{\rho,\rho}$ is dense in $BS^m_{\rho,\rho,N,N}$, the above mentioned endpoint results also hold if $BS^m_{\rho,\rho}$ is replaced with $BS^m_{\rho,\rho,N,N}$ for large enough N possibly depending on ρ and m. Lemma 7 and trilinear complex interpolation (see the book of Bergh and Löfström [7, Theorem 4.4.1]) then yield the thesis of the theorem for p_1 and p_2 such that $(\frac{1}{p_1}, \frac{1}{p_2})$ is on the border of the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

The result for p_1 and p_2 such that $(\frac{1}{p_1}, \frac{1}{p_2})$ is in the interior of the triangle follows by bilinear complex interpolation since, as shown in Figure 1, $m(p_1,p_2)$ is constant along horizontal segments in region I, $m(p_1,p_2)$ is constant along vertical segments in region II, $m(p_1,p_2)$ is constant along diagonal segments in region III and $m(p_1,p_2)$ is constant in region IV.

We now prove (ii) and (iii) (case $p<1$). Here we have to assume $\rho>0$. Theorem 10 yields

$$T : BS^m_{\rho,\rho} \times L^1 \times L^1 \to L^{\frac{1}{2},\infty} \quad \text{for } m < 2n(\rho-1) \text{ (point } (1,1) \text{ in Figure 1)},$$

which together with the boundedness properties at the points $(1,0)$ and $(0,1)$ in Figure 1 (as stated above), Lemma 7, and trilinear complex interpolation gives that

$$T : BS^m_{\rho,\rho} \times L^{p_1} \times L^{p_2} \to L^{p,\infty}, \quad m < m(p_1,p_2),$$

for $(\frac{1}{p_1}, \frac{1}{p_2})$ on the segments joining the points $(0,1)$ to $(1,1)$, $(1,0)$ to $(1,1)$, and $(\frac{1}{2}, \frac{1}{2})$ to $(1,1)$, in Figure 1. This gives Part (iii).

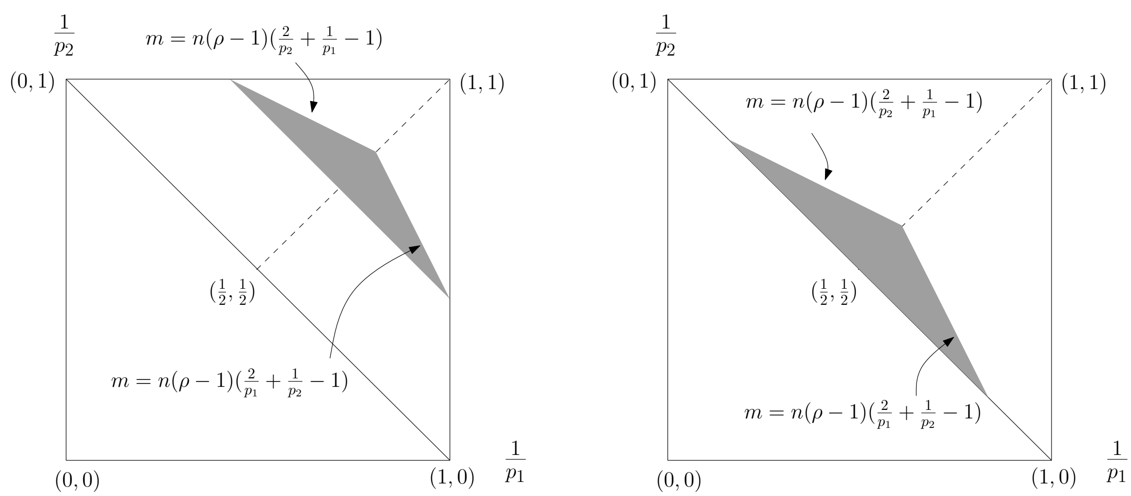

FIGURE 2. Case $p < 1$ of Theorem 2

For Part (ii) consider the shaded triangle as indicated in each case presented in Figure 2. The value $m(p_1,p_2)$ is constant, say m, on the upper border of this triangle, which is given by two segments with equations $m = n(\rho-1)(2/p_1 + 1/p_2 - 1)$ (inside triangle with vertices $(1,1)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1,0)$) and $m = n(\rho-1)(2/p_2+1/p_1-1)$ (inside triangle with vertices $(1,1)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1,0)$). Then Part (ii) follows by bilinear real interpolation using the weak type estimates obtained above for the vertices of the shaded triangle.

□

Remark 4.1. We note that the proof of Theorem 2 given for the case $p_1 = p_2 = p = \infty$ does not require any assumptions on the derivatives of the symbol σ in the space variables. This particular result is included in [24, Theorem 3.3], which yields boundedness properties in Lebesgue spaces of bilinear pseudo-differential operators with rough symbols in the space variables as a consequence of $|T_\sigma(f,g)|$ being pointwise bounded in terms of the Hardy-Littlewood maximal operator evaluated at f and g. For completeness, we have provided another proof of the case $p_1 = p_2 = \infty$ of Theorem 2 following the arguments of the corresponding linear result in [20].

5. PROOF OF THEOREM 3

In this section we continue to use L^p and $\|\cdot\|_{L^p}$ to denote the Lebesgue space $L^p(\mathbb{R}^n)$ and its norm, respectively. Sometimes it will be necessary to make explicit

the variable of integration, say integration with respect to x, in which case we employ the notation $\|\cdot\|_{L^p(dx)}$.

Proof of Theorem 3. Without loss of generality we may assume that the symbol σ has compact support in the frequency variables ξ and η. Otherwise, define $\sigma_\varepsilon(x,\xi,\eta) := \varphi(\varepsilon\,\xi,\varepsilon\,\eta)\sigma(x,\xi,\eta)$, where φ is a smooth function compactly supported in $B(0,1)$ such that $0\le\varphi\le 1$ and $\varphi(0,0)=1$. It easily follows that $C(\sigma_\varepsilon)\lesssim C(\sigma)$ and that $\lim_{\varepsilon\to 0} T_{\sigma_\varepsilon}(f,g) = T_\sigma(f,g)$ pointwise for f and g belonging to the class $\mathcal{U}$ of functions whose Fourier transforms are in $\mathcal{C}_0^\infty$. Assuming the result for symbols of compact support, by Fatou's lemma,

$$\|T_\sigma(f,g)\|_{L^2} \le \liminf_{\varepsilon\to 0} \|T_{\sigma_\varepsilon}(f,g)\|_{L^2} \le C(\sigma)\,\|f\|_{L^2}\|g\|_{L^2},$$

for $f,\, g\in\mathcal{U}$. Since $\mathcal{U}$ is dense in L^2 the desired result holds for non-compactly supported symbols as well.

Suppose first that σ is x-independent and define $\tau(\xi,\eta) := \sigma(x,\xi,\eta)$. Then

$$\widehat{T_\tau(f,g)}(\xi) = \int_{\mathbb{R}^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\tau(\xi-\eta,\eta)d\eta,$$

and the Cauchy-Schwarz inequality yields

$$|\widehat{T_\tau(f,g)}(\xi)| \lesssim \left(\int_{\mathbb{R}^n} |\hat{f}(\xi-\eta)|^2|\hat{g}(\eta)|^2\,d\eta\right)^{\frac{1}{2}} \sup_{\xi\in\mathbb{R}^n}\|\tau(\xi-\cdot,\cdot)\|_{L^2}.$$

Integrating in ξ and using Plancherel's theorem it follows that

$$\|T_\tau(f,g)\|_{L^2} \lesssim \|f\|_{L^2}\|g\|_{L^2} \sup_{\xi\in\mathbb{R}^n}\|\tau(\xi-\cdot,\cdot)\|_{L^2}, \tag{5.6}$$

which implies the desired result.

Next, we continue working with an x-independent symbol $\tau(\xi,\eta)$ in order to get estimates that will be useful later for x-dependent symbols. Let Φ be a smooth function compactly supported in $B(0,\sqrt{n})$ such that $0\le\Phi\le 1$ and

$$\sum_{k\in\mathbb{Z}^n}\Phi(k-x) = 1, \quad x\in\mathbb{R}^n.$$

For a function h defined in $\mathbb{R}^n$ and $l\in\mathbb{Z}^n$, we set $h_l(x) := \Phi(x-l)h(x)$. We will show that for every $N\in\mathbb{N}$

$$\|\Phi(\cdot-l)T_\tau(f,g)\|_{L^2} \lesssim \sup_{\substack{\xi\in\mathbb{R}^n\\ |\alpha|\le 2N}} \|\partial_\xi^\alpha\tau(\xi-\cdot,\cdot)\|_{L^2} \sum_{j,k\in\mathbb{Z}^n} \frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|+|l-k|)^N}, \tag{5.7}$$

for all $l\in\mathbb{Z}^n$ and with constants independent of τ, l, f and g.

We have

$$\Phi(x-l)T_\tau(f,g)(x) = \sum_{j,k\in\mathbb{Z}^n}\Phi(x-l)T_\tau(f_j,g_k)(x), \quad x\in\mathbb{R}^n,$$

and therefore (5.7) will follow from the estimate

$$\|\Phi(\cdot-l)T_\tau(f_j,g_k)\|_{L^2} \lesssim \sup_{\substack{\xi\in\mathbb{R}^n\\ |\alpha|\le 2N}} \|\partial_\xi^\alpha\tau(\xi-\cdot,\cdot)\|_{L^2} \frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|+|l-k|)^N}. \tag{5.8}$$

Fix $l \in \mathbb{Z}^n$. When $j,\, k \in \mathbb{Z}^n$ are such that $|l-j|+|l-k| \le 10$, we apply (5.6):

$$\begin{aligned}\|\Phi(\cdot - l)T_\tau(f_j,g_k)\|_{L^2} &\le \|T_\tau(f_j,g_k)\|_{L^2}\\ &\lesssim \sup_{\xi\in\mathbb{R}^n}\|\tau(\xi-\cdot,\cdot)\|_{L^2}\|f_j\|_{L^2}\|\ g_k\|_{L^2}\\ &\lesssim \sup_{\xi\in\mathbb{R}^n}\|\tau(\xi-\cdot,\cdot)\|_{L^2}\frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|+|l-k|)^N},\end{aligned}$$

for every integer N and therefore (5.8) holds.

We now consider j and k such that $|l-j|+|l-k| \ge 10$ and, without loss of generality, we assume that $|l-j| \ge |l-k|$. Then

$$\begin{aligned}T_\tau(f_j,g_k)(x) &= \int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^{2n}} e^{i(\xi\cdot(x-y)+\eta\cdot(x-z))}\tau(\xi,\eta)d\xi d\eta\right) f_j(y)g_k(z)dydz\\ &= \int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^{2n}} e^{i(\xi\cdot(x-y)+\eta\cdot(x-z))}(1-\Delta_\xi)^N\tau(\xi,\eta)d\xi d\eta\right)\frac{f_j(y)g_k(z)\,dydz}{(1+|x-y|^2)^N}\\ &= \int_{\mathbb{R}^{2n}}\mathcal{F}_{2n}((1-\Delta_\xi)^N\tau)(y-x,z-x)\frac{f_j(y)g_k(z)\,dydz}{(1+|x-y|^2)^N},\end{aligned}$$

where $\mathcal{F}_{2n}$ denotes the Fourier transform in $\mathbb{R}^{2n}$. Multiplying by $\Phi(x-l)$ and using the Sobolev embedding $W^{P,2}\subset L^\infty$ for any $P>n/2$, by fixing $x\in\mathbb{R}^n$, it follows that

$$\begin{aligned}&|\Phi(x-l)T_\tau(f_j,g_k)(x)|\\ &\lesssim \sup_{a\in\mathbb{R}^n}\left|\Phi(a-l)\int_{\mathbb{R}^{2n}}\mathcal{F}_{2n}((1-\Delta_\xi)^N\tau)(y-x,z-x)\frac{f_j(y)g_k(z)}{(1+|a-y|^2)^N}dydz\right|\\ &\lesssim \sup_{|\beta|\le P}\left\|\int_{\mathbb{R}^{2n}}\mathcal{F}_{2n}((1-\Delta_\xi)^N\tau)(y-x,z-x)\,(\partial_a^\beta\gamma_{l,N})(a,y)\,f_j(y)g_k(z)dydz\right\|_{L^2(da)},\end{aligned}$$

where $\gamma_{l,N}(a,y) := \frac{\Phi(a-l)}{(1+|a-y|^2)^N}$. Therefore,

$$\begin{aligned}&\|\Phi(\cdot-l)T_\tau(f_j,g_k)\|_{L^2}\\ &\lesssim \sup_{|\beta|\le P}\left\|\int_{\mathbb{R}^{2n}}\mathcal{F}_{2n}((1-\Delta_\xi)^N\tau)(y-x,z-x)\,(\partial_a^\beta\gamma_{l,N})(a,y)\,f_j(y)g_k(z)dydz\right\|_{L^2(dadx)}\\ &= \sup_{|\beta|\le P}\left\|T_{((1-\Delta_\xi)^N\tau)}\left(\partial_a^\beta\left(\frac{\Phi(a-l)}{(1+|a-\cdot|^2)^N}\right)f_j(\cdot),g_k(\cdot)\right)(x)\right\|_{L^2(dadx)}.\end{aligned}$$

Applying (5.6) to $T_{((1-\Delta_\xi)^N\tau)}$ then yields,

$$\begin{aligned}&\|\Phi(\cdot-l)T_\tau(f_j,g_k)\|_{L^2}\\ &\lesssim \sup_{\xi\in\mathbb{R}^n}\|(1-\Delta_\xi)^N\tau(\xi-\cdot,\cdot)\|_{L^2}\sup_{|\beta|\le P}\left\|\partial_a^\beta\left(\frac{\Phi(a-l)}{(1+|a-y|^2)^N}\right)f_j(y)\right\|_{L^2(dady)}\|g_k\|_{L^2}\\ &\lesssim \sup_{\xi\in\mathbb{R}^n}\|(1-\Delta_\xi)^N\tau(\xi-\cdot,\cdot)\|_{L^2}\frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|^2)^N},\end{aligned}$$

giving (5.8), where we have used that

$$\left\|\partial_a^\beta\left(\frac{\Phi(a-l)}{(1+|a-y|^2)^N}\right)\right\|_{L^2(da)} \lesssim \frac{1}{(1+|l-j|^2)^N}, \qquad y\in B(j,\sqrt{n}).$$

Consider now an x-dependent symbol. Then

$$T_\sigma(f,g)(x) = U_x(f,g)(x),$$

where

$$U_y(f,g)(x) := \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi+\eta)}\widehat{f}(\xi)\widehat{g}(\eta)\sigma(y,\xi,\eta)d\xi d\eta, \quad x,\, y\in\mathbb{R}^n.$$

Fixing $x\in\mathbb{R}^n$, $l\in\mathbb{Z}^n$, and using the Sobolev embedding $W^{s,2}\hookrightarrow L^\infty$ for an integer $s>n/2$, we get

$$\begin{aligned}|\Phi(x-l)T_\sigma(f,g)(x)| &\le \sup_{y\in\mathbb{R}^n}|\Phi(y-l)U_y(f,g)(x)| \\ &\le \sum_{|\beta|\le s}\|\partial_y^\beta\left(\Phi(y-l)U_y(f,g)(x)\right)\|_{L^2(dy)} \\ &\lesssim \sum_{|\beta|\le s}\|\chi_{B(l)}(y)\partial_y^\beta U_y(f,g)(x)\|_{L^2(dy)},\end{aligned}$$

where $B(l)=B(l,\sqrt{n})$. Let $\tilde\Phi$ be a smooth function supported in $B(0,\sqrt{n})$ such that $\tilde\Phi\Phi=\Phi$. Multiplying by $\tilde\Phi(x-l)$, integrating in x and using Fubini's Theorem, we obtain

$$\|\Phi(\cdot-l)T_\sigma(f,g)\|_{L^2} \lesssim \sum_{|\beta|\le s}\left\|\chi_{B(l)}(y)\left\|\chi_{B(l)}(x)\tilde\Phi(x-l)\partial_y^\beta U_y(f,g)(x)\right\|_{L^2(dx)}\right\|_{L^2(dy)}.$$

For each $\beta\in\mathbb{N}_0^n$, $|\beta|\le s$, and $y\in\mathbb{R}^n$, we look at $\partial_y^\beta U_y$ as the bilinear multiplier defined by the x-independent symbol

$$\tau_y^\beta(\xi,\eta) := \partial_y^\beta\sigma(y,\xi,\eta).$$

Then applying (5.7), which also holds if on its left hand side Φ is replaced by $\tilde\Phi$, we deduce

$$\begin{aligned}&\|\Phi(\cdot-l)T_\sigma(f,g)\|_{L^2} \\ &\lesssim \sum_{|\beta|\le s}\sum_{j,k\in\mathbb{Z}^n}\sup_{\substack{\xi\in\mathbb{R}^n\\|\alpha|\le 2N}}\sup_{y\in\mathbb{R}^n}\|\partial_\xi^\alpha\tau_y^\beta(\xi-\cdot,\cdot)\|_{L^2}\frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|+|l-k|)^N},\end{aligned}$$

which implies

$$\|\Phi(\cdot-l)T_\sigma(f,g)\|_{L^2} \lesssim C(\sigma)\sum_{j,k\in\mathbb{Z}^n}\frac{\|f_j\|_{L^2}\|g_k\|_{L^2}}{(1+|l-j|+|l-k|)^N},$$

with

$$C(\sigma) := \sup_{\substack{|\beta|\le s\\|\alpha|\le 2N}}\sup_{\xi,y\in\mathbb{R}^n}\|\partial_\xi^\alpha\partial_y^\beta\sigma(y,\xi-\cdot,\cdot)\|_{L^2}.$$

Using Hölder's inequality we then obtain that

$$\begin{aligned}&\|\Phi(\cdot-l)T_\sigma(f,g)\|_{L^2}^2\\&\lesssim C(\sigma)^2\sum_{j,k\in\mathbb{Z}^n}\frac{\|f_j\|_{L^2}^2\|g_k\|_{L^2}^2}{(1+|l-j|+|l-k|)^N}\sum_{j,k\in\mathbb{Z}^n}\frac{1}{(1+|l-j|+|l-k|)^N}.\end{aligned}$$

Choosing $N>2n$, the second sum on the right hand side is finite and after summing over $l\in\mathbb{Z}^n$, we conclude that

$$\sum_{l\in\mathbb{Z}^n}\|T_\sigma(f,g)_l\|_{L^2}^2=\sum_{l\in\mathbb{Z}^n}\|\Phi(\cdot-l)T_\sigma(f,g)\|_{L^2}^2\lesssim C(\sigma)^2\sum_{j\in\mathbb{Z}^n}\|f_j\|_{L^2}^2\sum_{k\in\mathbb{Z}^n}\|g_k\|_{L^2}^2.$$

The desired result follows by taking $N=2n+1$, $s=[\frac{n}{2}]+1$, and noting that $\|h\|_{L^2}^2\sim\sum_j\|h_j\|^2$. $\square$

6. Proof of Theorem 4

The following lemmas, whose proof are included in Section 8, will be used to prove Theorem 4.

Lemma 11. *Let $m\in\mathbb{R}$, $0\le\delta,\ \rho\le 1$, $\sigma\in BS^m_{\rho,\delta}$ and $N>n$.*
(a) If $0<R\le 1$ and $\operatorname{supp}(\sigma)\subset\{(x,\xi,\eta):|\xi|+|\eta|\le R\}$ then

$$\|T_\sigma(f,g)\|_{L^\infty}\lesssim R^{2n}\,\|\sigma\|_{0,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty}\,,\qquad f,\,g\in L^\infty.$$

(b) If $R\ge 1$ and $\operatorname{supp}(\sigma)\subset\{R\le|\xi|+|\eta|\le 4R\}$ then

$$\|T_\sigma(f,g)\|_{L^\infty}\lesssim R^{(1-\rho)n+m}\,\|\sigma\|_{0,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty}\,,\qquad f,\,g\in L^\infty.$$

Lemma 12. *Let $Q\subset\mathbb{R}^n$ be a cube with diameter d and $\sigma\in BS^m_{\rho,\delta}$ with $m=n(\rho-1)$, $0\le\delta,\ \rho\le 1$, such that*

$$\operatorname{supp}(\sigma)\subset\{(x,\xi,\eta):|\xi|+|\eta|\le d^{-1}\}.$$

Then, for every $N>n$,

$$\frac{1}{|Q|}\int|T_\sigma(f,g)(x)-T_\sigma(f,g)_Q|\,dx\lesssim\|\sigma\|_{1,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty}\,,\qquad f,\,g\in\mathcal{S},$$

with constants only depending on n, N, ρ and δ. Here $T_\sigma(f,g)_Q$ is the average of $T_\sigma(f,g)$ over Q.

Lemma 13. *Let $d>0$ and $\sigma\in BS^m_{\rho,\delta}$, $m=n(\rho-1)$, $0\le\delta,\rho\le 1$, such that*

$$\operatorname{supp}(\sigma)\subset\{(x,\xi,\eta):|\xi|+|\eta|\ge d^{-1}\}.$$

Let $\phi\in\mathcal{S}$, $\phi\ge 0$, and $\operatorname{supp}(\hat\phi)\subset\{z\in\mathbb{R}^n:|z|\le\frac{1}{8}d^{-\rho}\}$. For $f,g\in\mathcal{S}$, define

$$R(f,g)(x):=\phi^2(x)T_\sigma(f,g)(x)-T_\sigma(\phi f,\phi g)(x).$$

Then, for every $N>n$, we have

$$\|R(f,g)\|_{L^\infty}\lesssim\|\sigma\|_{0,2N+1}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty}\,,\qquad f,\,g\in\mathcal{S},$$

with constants only depending on n, N, ρ and δ.

Remark 6.1. The proofs of the above lemmas show that $BS^m_{\rho,\delta}$ can be replaced by $BS^m_{\rho,\delta,0,2N}$, $BS^m_{\rho,\delta,1,2N}$ and $BS^m_{\rho,\delta,0,2N+1}$, respectively (see definition of these spaces in Section 2.1).

Proof of Theorem 4. Given $\sigma \in BS^m_{\rho,0}$, with $m = n(\rho-1)$, we have to prove that

$$\frac{1}{|Q|}\int |T_\sigma(f,g)(x) - T_\sigma(f,g)_Q|\,dx \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty}, \tag{6.9}$$

for all cubes $Q \subset \mathbb{R}^n$ and $f,\, g \in \mathcal{S}$.

Let Q be a cube with diameter d and assume first that $d \leq 1$. We write

$$\sigma(x,\xi,\eta) = \sigma(x,\xi,\eta)(1-\theta(\xi,\eta)) + \sigma(x,\xi,\eta)\theta(\xi,\eta) =: \sigma_1 + \sigma_2,$$

where $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a smooth, non-negative function, $\theta(\xi,\eta) = \tilde{\theta}(d\,\xi, d\,\eta)$ with

$$\operatorname{supp}(\tilde{\theta}) \subset \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi| + |\eta| \geq 1\},$$

and $\tilde{\theta} \equiv 1$ in $\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi| + |\eta| \geq 2\}$. Since $d \leq 1$, then $\sigma_1,\, \sigma_2 \in BS^m_{\rho,0}$ satisfy

$$\|\sigma_j\|_{K,M} \lesssim \|\sigma\|_{K,N}, \quad K, M \in \mathbb{N}_0, j = 1,2, \tag{6.10}$$

with constants independent of d and σ.

Let ϕ be as in Lemma 13, this is, $\phi \in \mathcal{S}$, $\phi \geq 0$, and $\operatorname{supp}(\hat{\phi}) \subset \{z \in \mathbb{R}^n : |z| \leq d^{-\rho}/8\}$. In addition, we assume $\phi \equiv 1$ on Q and, in accordance with the uncertainty principle, we choose ϕ such that $\|\phi\|_{L^2} \lesssim d^{\frac{n\rho}{2}}$. For $x \in Q$ we have

$$\begin{aligned} T_\sigma(f,g)(x) &= T_{\sigma_1}(f,g)(x) + T_{\sigma_2}(f,g)(x) = T_{\sigma_1}(f,g)(x) + \phi^2(x)T_{\sigma_2}(f,g)(x) \\ &= T_{\sigma_1}(f,g)(x) + T_{\sigma_2}(\phi f, \phi g)(x) + R(f,g)(x), \end{aligned}$$

where $R(f,g)(x) = \phi^2(x)T_{\sigma_2}(f,g)(x) - T_{\sigma_2}(\phi f,\phi g)(x)$.

In order to get (6.9), it is enough to prove the inequality

$$\|T_{\sigma_2}(\phi f, \phi g)\|_{L^1(Q)} \lesssim \|\sigma_2\|_{K,M}\, d^n \|f\|_{L^\infty}\|g\|_{L^\infty}, \quad f, g \in \mathcal{S}, \tag{6.11}$$

for some $K,\, M \in \mathbb{N}_0$. Indeed, using (6.11) and Lemmas 12 and 13, for $N > n$, we write

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |T_\sigma(f,g)(x) - T_\sigma(f,g)_Q|\,dx \\ &\leq \frac{1}{|Q|}\int_Q |T_{\sigma_1}(f,g)(x) - T_{\sigma_1}(f,g)_Q|\,dx + \frac{2}{|Q|}\|T_{\sigma_2}(\phi f,\phi g)\|_{L^1(Q)} + 2\|R(f,g)\|_{L^\infty} \\ &\lesssim \left(\|\sigma_1\|_{1,2N} + \|\sigma_2\|_{K,M} + \|\sigma_2\|_{0,2N+1}\right)\|f\|_{L^\infty}\|g\|_{L^\infty}, \end{aligned}$$

and therefore (6.9) holds when the diameter of Q is less than or equal to 1.

In turn, (6.11) will follow from

$$\|T_{\sigma_2}(\phi f,\phi g)\|_{L^2} \lesssim \|\sigma_2\|_{K,M}\, d^{\frac{n}{2}}\|f\|_{L^\infty}\|g\|_{L^\infty}, \quad f, g \in \mathcal{S}, \tag{6.12}$$

since

$$\frac{1}{|Q|}\|T_{\sigma_2}(\phi f,\phi g)\|_{L^1(Q)} \leq \frac{1}{|Q|^{1/2}}\|T_{\sigma_2}(\phi f,\phi g)\|_{L^2(Q)} \leq \frac{1}{|Q|^{1/2}}\|T_{\sigma_2}(\phi f,\phi g)\|_{L^2}.$$

Moreover, because ϕ satisfies $\|\phi\|_{L^2} \lesssim d^{\frac{\rho n}{2}}$, (6.12) can be reduced to proving that

$$\|T_{\sigma_2}\|_{L^2\times L^2\to L^2} \lesssim \|\sigma_2\|_{K,M}\, d^{\frac{n}{2}-\rho n}. \tag{6.13}$$

By Theorem 3, the support of σ_2, and the fact that $\sigma_2 \in BS^m_{\rho,0}$ with $m = n(\rho-1)$ and $0<\rho<\frac{1}{2}$, we obtain

$$\begin{aligned}
\|T_{\sigma_2}\|_{L^2\times L^2\to L^2} &\lesssim \sup_{\substack{|\beta|\le[\frac{n}{2}]+1\\ |\alpha|\le 2(2n+1)}} \sup_{y,\xi\in\mathbb{R}^n} \|\partial_\xi^\alpha \partial_y^\beta \sigma_2(y,\xi-\cdot,\cdot)\|_{L^2}\\
&\lesssim \sup_{\xi\in\mathbb{R}^n} \|\chi_{\{|\xi-\eta|+|\eta|\ge d^{-1}\}}(\xi,\eta)\,(1+|\xi-\eta|+|\eta|)^m\,\|_{L^2(d\eta)}\\
&\lesssim \|\sigma_2\|_{K,M}\left(\int_{|\eta|\ge d^{-1}} |\eta|^{2m} d\eta\right)^{1/2} + \left(\int_{|\eta|\le d^{-1}} d^{-2m} d\eta\right)^{1/2}\\
&\lesssim \|\sigma_2\|_{K,M}\, d^{-m-\frac{n}{2}} = \|\sigma_2\|_{K,M}\, d^{\frac{n}{2}-\rho n},
\end{aligned}$$

where we have taken $K=[\frac{n}{2}]+1$ and $M=2(2n+1)$.

The case $d>1$ follows using the decomposition of σ with $\theta=\tilde{\theta}$ and then proceeding analogously but applying to the term corresponding to T_{σ_1} Lemma 11 instead of Lemma 12. $\square$

7. Proof of Theorem 5

For $s>0$, we recall the bilinear fractional integral operator of order $s>0$, introduced in Kenig-Stein [21], defined by

$$\mathcal{I}_s(f,g)(x) := \int_{\mathbb{R}^{2n}} \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-s}}\,dydz, \quad x\in\mathbb{R}^n. \tag{7.14}$$

It easily follows that

$$\mathcal{I}_s(f,g)(x) \le I_{s_1}(f)(x)\, I_{s_2}(g)(x), \qquad x\in\mathbb{R}^n,\ s_1+s_2=s,$$

where

$$I_\tau(h)(x) = \int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-\tau}}\,dy, \quad 0<\tau<n,$$

is the linear fractional integral. The boundedness properties of I_τ, $0<\tau<n$, and Hölder's inequality imply that $\mathcal{I}_s$ is bounded form $L^{p_1}\times L^{p_2}$ into L^p with $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}-\frac{s}{n}$, $0<s<2n$, $1<p_1,\ p_2<\infty$, $q>0$.

We now observe that if $\sigma\in BS^m_{\rho,\delta}$, $m\le 2n(\rho-1)-\rho s$, $0<s<2n$, then part (v) of Theorem E implies that

$$|T_\sigma(f,g)(x)| \lesssim |\mathcal{I}_s(f,g)(x)|. \tag{7.15}$$

Therefore Theorem 5 follows from this inequality and the boundedness properties of $\mathcal{I}_s$. The case $\rho=1$ of Theorem 5 was treated in [8].

8. Proof of lemmas from Section 6

Proof of Lemma 11. We have

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \mathcal{K}(x, x-y, x-z) f(y) g(z)\, dy\, dz, \tag{8.16}$$

where

$$\mathcal{K}(x,y,z) = \int_{\mathbb{R}^{2n}} e^{i\xi\cdot y} e^{i\eta\cdot z} \sigma(x,\xi,\eta)\, d\xi\, d\eta = \mathcal{F}_{2n}^{-1}(\sigma(x,\cdot,\cdot))(y,z),$$

and $\mathcal{F}_{2n}$ denotes the inverse Fourier transform in $\mathbb{R}^{2n}$. Then, it is enough to show that for $N > n$, $N \in \mathbb{N}_0$,

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^{2n}} |\mathcal{K}(x,y,z)|\, dy\, dz \lesssim R^{2n} \, \|\sigma\|_{0,2N}. \tag{8.17}$$

and

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^{2n}} |\mathcal{K}(x,y,z)|\, dy\, dz \lesssim R^{(1-\rho)n+m} \, \|\sigma\|_{0,2N}. \tag{8.18}$$

for part (a) and part (b), respectively. (Note that this allows to extend T_σ to a bounded operator form $L^\infty \times L^\infty$ into L^∞ by using the representation (8.16) to define $T_\sigma(f,g)$ for $f,g \in L^\infty$).

Since σ is a smooth function with compact support in ξ and η we have

$$\begin{aligned} (1+|(y,z)|^2)^N \mathcal{K}(x,y,z) &= \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\, (1-\Delta_\xi-\Delta_\eta)^N (e^{i\xi\cdot y}\, e^{i\eta\cdot z})\, d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}} (1-\Delta_\xi-\Delta_\eta)^N (\sigma(x,\xi,\eta))\, e^{i\xi\cdot y}\, e^{i\eta\cdot z}\, d\xi d\eta \\ &= \mathcal{F}_{2n}^{-1}((1-\Delta_\xi-\Delta_\eta)^N(\sigma(x,\cdot,\cdot)))(y,z), \end{aligned} \tag{8.19}$$

and similarly,

$$|(y,z)|^{2N}\mathcal{K}(x,y,z) = \mathcal{F}_{2n}^{-1}((-\Delta_\xi-\Delta_\eta)^N(\sigma(x,\cdot,\cdot)))(y,z). \tag{8.20}$$

For part (a), we use (8.19) and that $R \le 1$ to get,

$$|\mathcal{K}(x,y,z)| \lesssim \frac{R^{2n}\, \|\sigma\|_{0,2N}}{(1+|(y,z)|^2)^N}$$

and then (8.17) follows since $N > n$.

For part (b) we write

$$\int_{\mathbb{R}^{2n}} |\mathcal{K}(x,y,z)|\, dydz = \int\limits_{|y|+|z|\le R^{-\rho}} |\mathcal{K}(x,y,z)|\, dydz + \int\limits_{|y|+|z|\ge R^{-\rho}} |\mathcal{K}(x,y,z)|\, dydz.$$

Let us now estimate the first integral. By Cauchy-Schwarz inequality, Plancherel's identity and the fact that $R \geq 1$, we have

$$\begin{aligned}\left(\int\limits_{|y|+|z|\leq R^{-\rho}} |\mathcal{K}(x,y,z)|\,dydz\right)^2 &\lesssim R^{-2\rho n} \int\limits_{|y|+|z|\leq R^{-\rho}} |\mathcal{K}(x,y,z)|^2\,dydz \\ &\lesssim R^{-2\rho n} \int\limits_{|\xi|+|\eta|\sim R} |\sigma(x,\xi,\eta)|^2\,d\xi d\eta \\ &\lesssim \|\sigma\|_{0,0}^2\, R^{-2\rho n} \int\limits_{|\xi|+|\eta|\sim R} (1+|\xi|+|\eta|)^{2m} d\xi d\eta \\ &\lesssim \|\sigma\|_{0,0}^2\, R^{-2\rho n} R^{2m+2n} = \|\sigma\|_{0,0}^2\, R^{2((1-\rho)n+m)}.\end{aligned}$$

Next, we estimate the second integral. Multiplying and dividing by $|(y,z)|^{2N}$, and using the Cauchy-Schwarz inequality, that $N > n$, (8.20), Plancherel's identity, and that $R \geq 1$, it follows that

$$\begin{aligned}\left(\int\limits_{|y|+|z|\geq R^{-\rho}} |\mathcal{K}(x,y,z)|\,dydz\right)^2 &\lesssim \left(\int\limits_{|y|+|z|\geq R^{-\rho}} \frac{1}{|(y,z)|^{4N}} dydz\right) \\ &\quad\times \left(\int\limits_{|y|+|z|\geq R^{-\rho}} ||(y,z)|^{2N}\mathcal{K}(x,y,z)|^2 dydz\right) \\ &\lesssim R^{\rho(4N-2n)} \int\limits_{|\xi|+|\eta|\sim R} |(-\Delta_\xi - \Delta_\eta)^N \sigma(x,\xi,\eta))|^2\,d\xi d\eta \\ &\lesssim \|\sigma\|_{0,2N}^2\, R^{\rho(4N-2n)} \int\limits_{|\xi|+|\eta|\sim R} (1+|\xi|+|\eta|)^{2(m-\rho 2N)} d\xi d\eta \\ &\lesssim \|\sigma\|_{0,2N}^2\, R^{\rho(4N-2n)} R^{2(m-\rho 2N+n)} \\ &= \|\sigma\|_{0,2N}^2\, R^{2((1-\rho)n+m)}.\end{aligned}$$

The last two computations give (8.18). □

Proof of Lemma 12. Let Q, d, N, m and σ be as in the hypothesis. By definition,

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\, e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta, \quad f,\, g \in \mathcal{S}.$$

Hence, for a fixed $j = 1,\ldots,n$, the bilinear symbol $\tau = \tau(x,\xi,\eta)$ of the bilinear operator $\frac{\partial T_\sigma(f,g)}{\partial x_j}$ is given by

$$\tau(x,\xi,\eta) = i(\xi_j+\eta_j)\sigma(x,\xi,\eta) + \frac{\partial \sigma}{\partial x_j}(x,\xi,\eta).$$

Then symbol τ is also supported in $\{(x,\xi,\eta) : |\xi|+|\eta| \le d^{-1}\}$ and $\tau \in BS^{m+\delta}_{\rho,\delta}$. Elementary computations show that for K, $M \in \mathbb{N}_0$,

$$\|\tau\|_{K,M} \le \max(1,d^{-1})\,\|\sigma\|_{K+1,M}\,, \tag{8.21}$$

where $\|\tau\|_{K,M}$ corresponds to a norm of τ as an element of $BS^{m+\delta}_{\rho,\delta}$, while $\|\sigma\|_{K+1,M}$ corresponds to a norm of σ as an element of $BS^{m}_{\rho,\delta}$. Then

$$\begin{aligned}\int_Q |T_\sigma(f,g)(x) - T_\sigma(f,g)_Q|\,dx &= \frac{1}{|Q|}\int_Q \left|\int_Q \left(T_\sigma(f,g)(x) - T_\sigma(f,g)(y)\right)\,dy\right|\,dx \\ &\le d\,|Q|\,\|\nabla T_\sigma(f,g)\|_{L^\infty} \lesssim d\,|Q|\,\|T_\tau(f,g)\|_{L^\infty} \\ &\lesssim d\,|Q|\,\min(1,d^{-2n})\,\|\tau\|_{0,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty} \\ &\lesssim |Q|\,\|\sigma\|_{1,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty} \quad \text{(by (8.21))},\end{aligned}$$

where we have used Lemma 11. The result follows.

$\square$

Proof of Lemma 13. Let d, N, m, ϕ and σ be as in the hypothesis. We notice that the bilinear symbol $\theta(x,\xi,\eta)$ of R is given by

$$\theta(x,\xi,\eta) = \int_{\mathbb{R}^{2n}} e^{ix\cdot(y+z)}\left(\sigma(x,\xi,\eta) - \sigma(x,\xi+y,\eta+z)\right)\hat{\phi}(y)\hat{\phi}(z)\,dy\,dz.$$

We first assume that $d \le 1$ and note that $\operatorname{supp}(\theta) \subset \{(x,\xi,\eta) : |\xi|+|\eta| \ge \frac{1}{2}d^{-1}\}$. Consider a partition of unity of $\mathbb{R}^{2n}$ given by $\{\psi_k\}_{k\in\mathbb{N}_0}$,

$$\sum_{k\ge 0}\psi_k(\xi,\eta) = 1, \quad \xi,\eta \in \mathbb{R}^n,$$

where $\psi_0 \in \mathcal{S}(\mathbb{R}^{2n})$ is supported in the set $\{(\xi,\eta) : |\xi|+|\eta| \le 2d^{-1}\}$ and $\psi_k(\xi,\eta) = \psi(d2^{-k}\xi, d2^{-k}\eta)$ with $\psi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\operatorname{supp}(\psi) \subset \{(\xi,\eta) : 1/2 \le |\xi|+|\eta| \le 2\}$ for $k \ge 1$. Then $\operatorname{supp}(\psi_k) \subset \{(\xi,\eta) : |\xi|+|\eta| \sim 2^k d^{-1}\}$ for $k \ge 1$ and

$$\theta(x,\xi,\eta) = \sum_{k\ge 0}\theta_k(x,\xi,\eta),$$

where $\theta_k(x,\xi,\eta) := \theta(x,\xi,\eta)\psi_k(\xi,\eta)$. We will show that for all integers $M, k \ge 0$

$$\|\theta_k\|_{0,M} \lesssim 2^{-\rho k}\,\|\sigma\|_{0,M+1}\,, \tag{8.22}$$

with constants depending only on M, n, ρ, and δ.

Define R_k as the bilinear pseudo-differential operator with kernel θ_k. The lemma will follow from (8.22). Indeed,

$$\begin{aligned}\|R(f,g)\|_{L^\infty} &\le \sum_{k\ge 0}\|R_k(f,g)\|_{L^\infty} \lesssim \sum_{k\ge 0}\|\theta_k\|_{0,2N}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty} \quad \text{(by Lemma 11)} \\ &\lesssim \sum_{k\ge 0} 2^{-\rho k}\,\|\sigma\|_{0,2N+1}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty} \quad \text{(by (8.22))} \\ &\lesssim \|\sigma\|_{0,2N+1}\,\|f\|_{L^\infty}\,\|g\|_{L^\infty}\,.\end{aligned}$$

To prove (8.22), consider multi-indices β and γ such that $|\beta|, |\gamma| \le M$. Since

$$\theta_k(x,\xi,\eta) = \int_{\mathbb{R}^{2n}} e^{ix\cdot(y+z)}\psi_k(\xi,\eta)\left(\sigma(x,\xi,\eta) - \sigma(x,\xi+y,\eta+z)\right)\hat{\phi}(y)\hat{\phi}(z)\,dy\,dz,$$

we have

$$\left(\partial_\xi^\beta \partial_\eta^\gamma \theta_k\right)(x,\xi,\eta) = \sum_{\lambda\le\gamma,\omega\le\beta} C_{\beta,\gamma,\omega,\lambda}\left(\partial_\xi^{\beta-\omega}\partial_\eta^{\gamma-\lambda}\psi\right)(d2^{-k}\xi, d2^{-k}\eta)(2^{-k}d)^{|\gamma-\lambda|+|\beta-\omega|}$$

$$\times \int_{\mathbb{R}^{2n}} \hat{\phi}(y)\hat{\phi}(z)e^{ix\cdot(y+z)}\left((\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi,\eta) - (\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi+y,\eta+z)\right)\,dy\,dz.$$

The mean value theorem gives

$$(\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi,\eta) - (\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi+y,\eta+z) = (\nabla_\xi\partial_\xi^\omega\nabla_\eta\partial_\eta^\lambda\sigma)(x,\tilde{\xi},\tilde{\eta})\cdot(y,z),$$

where $(\tilde{\xi},\tilde{\eta}) = (\xi,\eta) + s\,(y,z)$ for some $s \in (0,1)$. Since $\sigma \in BS^m_{\rho,\delta}$, for $(\xi,\eta) \in \operatorname{supp}(\psi_k)\cap\operatorname{supp}(\theta)$ and $y,\, z \in \operatorname{supp}(\hat{\phi})$, we then have

$$\begin{aligned}\big|(\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi,\eta) &- (\partial_\xi^\omega\partial_\eta^\lambda\sigma)(x,\xi+y,\eta+z)\big| \\ &\lesssim \|\sigma\|_{0,M+1}\,(1+|\tilde{\xi}|+|\tilde{\eta}|)^{m-\rho(|\omega|+|\lambda|+1)}|(y,z)| \\ &\lesssim \|\sigma\|_{0,M+1}\,(1+|\xi|+|\eta|)^{m-\rho(|\omega|+|\lambda|+1)}|(y,z)|,\end{aligned}$$

where we have used that $|\tilde{\xi}|+|\tilde{\eta}| \simeq |\xi|+|\eta|$, since $|\xi|+|\eta| \simeq 2^k d^{-1}$ and $|y|+|z| \le d^{-\rho}/4 \le d^{-1}/4$. Putting all together, and using again that $2^k d^{-1} \ge 1$, $d \le 1$, and $1+|\xi|+|\eta| \simeq |\xi|+|\eta| \simeq 2^k d^{-1}$,

$$\begin{aligned}|\partial_\xi^\beta\partial_\eta^\gamma\theta_k(x,\xi,\eta)| &\lesssim \|\sigma\|_{0,M+1}\,(1+|\xi|+|\eta|)^{m-\rho(|\gamma|+|\beta|)}(1+2^k d^{-1})^{-\rho} \\ &\quad\times \sum_{\lambda\le\gamma,\,\omega\le\beta}(2^{-k}d)^{(1-\rho)(|\gamma-\lambda|+|\beta-\omega|)} \\ &\lesssim \|\sigma\|_{0,M+1}\,(1+|\xi|+|\eta|)^{m-\rho(|\gamma|+|\beta|)}2^{-\rho k},\end{aligned}$$

which gives (8.22). □

If $d > 1$ then we split θ as

$$\theta = \theta_1 + \theta_2,$$

where $\operatorname{supp}(\theta_1) \subset \{(\xi,\eta) : |\xi|+|\eta| \le 2\}$ (note that in the case $d > 1$ we also have $|y|,|z| \le d^{-\rho}/8 \le 1/8$) and $\operatorname{supp}(\theta_2) \subset \{(\xi,\eta) : |\xi|+|\eta| \ge 1\}$. A similar reasoning as above shows that $\|\theta_1\|_{0,M} \lesssim \|\sigma\|_{0,M+1}$. We then apply Lemma 11 to the bilinear pseudo-differential operator with symbol θ_1 and reduce the analysis of θ_2 to the case $d = 1$.

9. Weighted results

Given a weight w defined on $\mathbb{R}^n$ and $p > 0$, the notation L^p_w will be used to refer to the weighted Lebesgue space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\|f\|_{L^p_w} := \int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx < \infty$, when $w \equiv 1$ we will continue to simply write L^p and $\|f\|_{L^p}$, respectively.

If w_1, w_2 are weights defined on $\mathbb{R}^n$, $1 \le p_1, p_2 < \infty$, $q > 0$, and $w := w_1^{q/p_1} w_2^{q/p_2}$, we say that (w_1, w_2) satisfies the $A_{(p_1,p_2),q}$ condition (or that (w_1, w_2) belongs to the bilinear Muckenhoupt class $A_{(p_1,p_2),q}$) if

$$[(w_1, w_2)]_{A_{(p_1,p_2),q}} := \sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right) \prod_{j=1}^{2} \left(\frac{1}{|B|}\int_B w_j(x)^{1-p_j'}\,dx\right)^{\frac{q}{p_j'}} < \infty,$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$; when $p_j = 1$ $\left(\frac{1}{|B|}\int_B w_j(x)^{1-p_j'}\,dx\right)^{\frac{1}{p_j'}}$ is understood as $(\inf_B w_j)^{-1}$.

The classes $A_{(p_1,p_2),q}$ are inspired in the classes of weights $A_{p,q}$, $1 \le p, q < \infty$, defined by Muckenhoupt and Wheeden in [26] to study weighted norm inequalities for the fractional integral: a weight u defined on $\mathbb{R}^n$ is in the class $A_{p,q}$ if

$$\sup_B \left(\frac{1}{|B|}\int_B u^{\frac{q}{p}}\,dx\right)\left(\frac{1}{|B|}\int_B u^{(1-p')}\,dx\right)^{\frac{q}{p'}} < \infty.$$

The classes $A_{(p_1,p_2),p}$[1] for $1/p = 1/p_1 + 1/p_2$ were introduced by Lerner et al in [22] to study characterizations of weights for boundedness properties of certain bilinear maximal functions and bilinear Calderón-Zygmund operators in weighted Lebesgue spaces. Likewise, as shown by Moen [25], the classes $A_{(p_1,p_2),q}$ characterize the weights rendering analogous bounds for bilinear fractional integral operators .

Theorem 10 and [22, Corollary 3.9] imply the following result.

Corollary 14. *Let $0 \le \delta \le \rho \le 1$, $\delta < 1$, $0 < \rho$, $m_{cz} = 2n(\rho - 1)$, $1 \le p_1, p_2 < \infty$ and p given by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Suppose $\sigma \in BS^m_{\rho,\delta}$, $m < m_{cz}$, (w_1, w_2) satisfies the $A_{(p_1,p_2),p}$ condition and $w = w_1^{p/p_1} w_2^{p/p_2}$.*
(a) If $1 < p_1, p_2 < \infty$ then there exists $K, N \in \mathbb{N}_0$ such that

$$\|T_\sigma(f, g)\|_{L^p_w} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}_{w_1}} \|g\|_{L^{p_2}_{w_2}}.$$

(b) If $1 \le p_1, p_2 < \infty$ and $p_1 = 1$ or $p_2 = 1$ then there exists $K, N \in \mathbb{N}_0$ such that

$$\|T(f, g)\|_{L^{p,\infty}_w} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}_{w_1}} \|g\|_{L^{p_2}_{w_2}}.$$

Inequality (7.15) and [25, Theorem 3.5] yield the following:

Corollary 15 (Weighted version of Theorem 5)**.** *Let $0 \le \delta \le 1$, $0 < \rho \le 1$, $s \in (0, 2n)$, and $m_s := 2n(\rho - 1) - \rho s$. If $\sigma \in BS^m_{\rho,\delta}$, $m \le m_s$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_1} - \frac{s}{n}$, $1 < p_1, p_2 < \infty$, then there exist nonnegative integers K and N such that*

$$\|T_\sigma(f, g)\|_{L^q_w} \lesssim \|\sigma\|_{K,N}\, \|f\|_{L^{p_1}_{w_1}} \|g\|_{L^{p_2}_{w_2}},$$

for $w := w_1^{q/p_1} w_2^{q/p_2}$ and pairs of weights (w_1, w_2) satisfying the $A_{(p_1,p_2),q}$ condition.

[1] These classes were denoted by $A_{\vec{P}}$ in [22], with $\vec{P} = (p_1, p_2)$ determining $1/p = 1/p_1 + 1/p_2$.

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